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NONLINEAR MULTI-POINT PROBLEMS VIA CHELYSHKOV APPROACH

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ABSTRACT

This study addresses a numerical method based on Chelyshkov polynomials for solving nonlinear multi-point boundary value problems (NBVPs). These polynomials are orthogonal on [0,1], providing a solid basis for approximating solutions. The method converts the nonlinear equations with boundary conditions into a matrix structure and allows solutions to be computed in an efficient manner. The chosen examples demonstrate the method's accuracy and practical advantages over existing techniques. Beyond NBVPs, the approach can also be applied to a broader class of functional differential equations in fields such as physics, biology, and engineering, offering a robust and adaptable tool for researchers tackling complex problems.

1. INTRODUCTION

There has been a recent surge in interest in linear and nonlinear multi-point differential equations. Many of these nonlinear boundary value problems (NBVPs) encountered in engineering, applied mathematics, and various branches of physics cannot be solved analytically; obtaining closed-form solutions is challenging. However, the complexity does not stop there: numerous correctly formulated nonlinear boundary value problems either lack solutions or have multiple solutions. Despite these challenges, there is a growing demand among engineers and physicists to devise solution methods for these problems [1-4].

Presently, a plethora of numerical methods has emerged to address boundary value problems across a broad spectrum of phenomena in applied sciences, engineering, and physics, spanning areas like astrophysics, fluid dynamics, elasto-plastic problems, mathematical modeling of AFTI-F16 fighters, nonlinear vibration of electric machines, among others [5-10]. Some well-known approaches for solving such problems include the shooting method, Adomian decomposition method, variational iteration method, homotopy perturbation method, reproducing kernel method, spline techniques, finite difference method, Sinc-collocation method, and Pade approximation [11–19], among others.

In this paper, we focus on nonlinear BVPs featuring variable coefficients, formulated as follows [20], and aim to provide a concise overview of the methods and approaches employed in their solution in $\nu \in [a,b]$

$$\sum_{h=0}^{s_1}\sum_{j=0}^{s_2} P_{hj}(\nu)\gamma^{j}(\nu)\gamma^{(h)}(\nu) + \sum_{r=1}^{s_3}\sum_{s=1}^{s_4} O_{rs}(\nu)\gamma^{(s)}(\nu)\gamma^{(r)}(\nu) = h(\nu), \tag{1}$$

subject to the boundary conditions

$$\sum_{h=0}^{\omega-1}\left\{a_{ih}\gamma^{(h)}(0) + \sum_{k=0}^{K} c_{ik}\gamma^{(h)}(\nu_k) + b_{ih}\gamma^{(h)}(1)\right\} = \beta_i,\ \nu_k \in [0,1]. \tag{2}$$

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Here, $a_{ih}, b_{ih}, c_{ik}, c_k$ and β_i stand as real coefficients, with $y(\nu)$ representing an indeterminate function. The known functions $P_{hj}(\nu)$ and $O_{rs}(\nu)$ are characterized as continuous functions defined over the interval $[0,1]$.

The main goal of this paper is to find an approximate solution to Equation (1) that satisfies the boundary conditions given in Equation (2), using a truncated Chelyshkov series representation, denoted as

$$\gamma(\nu) \cong \sum_{k=0}^{\omega} a_k C_{k0}(\nu), \tag{3}$$

or briefly referred to as

$$\gamma_{\omega}(\nu) = \sum_{k=0}^{\omega} a_k C_k(\nu) \tag{4}$$

Here, the unknowns a_s, s $= 0,1,2,\ldots,\omega$ are the Chelyshkov coefficients. The parameter ω is chosen as a positive integer that meets the specified conditions. The Chelyshkov polynomials $C_k(\nu)$, $k = 0,1,2...,\omega$ defined in [21], are orthogonal over the interval $[0,1]$ with respect to the weight function 1 and can be explicitly expressed as

$$C_{\omega\lambda}(\nu) = \sum_{j=0}^{\omega-\lambda} (-1)^j \binom{\omega-\lambda}{j} \binom{\omega+\lambda+j+1}{\omega-\lambda} \nu^{\lambda+j},$$

These polynomials, exhibiting properties akin to classical orthogonal polynomials, serve as examples of alternative orthogonal polynomials. Additionally, they can be related to the Jacobi polynomials by the following relation.

$$C_{\omega\lambda}(\nu) = (-1)^{\omega-\lambda} \nu^{\eta-\lambda} P_{\omega-\lambda}^{(0,2\lambda+1)}(2\nu-1),\ \lambda = 0,1\ldots\omega.$$

Further exploration of these polynomials reveals that each member within the family of orthogonal polynomials has a degree ω with $\omega-\lambda$ distinct real roots. Consequently, for every ω, the polynomial possesses exactly ω simple roots within the interval $[0,1]$. Consequently, when the roots of the polynomial are selected as collocation points, accurate numerical quadrature can be attained [21-26].

In this study, Chelyshkov polynomials are chosen because they fit naturally into the matrix-based collocation framework. Their representation allows the algebraic system to be written in a straightforward matrix form, unlike some classical orthogonal polynomials such as Legendre or Chebyshev polynomials. This makes the implementation more manageable and enables the boundary conditions to be incorporated directly into the formulation. For nonlinear multi-point boundary value problems, this structural simplicity is one of the main reasons for preferring Chelyshkov polynomials in the present work.

Based on this approach, a collocation procedure is constructed to obtain approximate solutions to the problems under consideration. The differential equation is transformed into a nonlinear algebraic system, and the effectiveness of the method is illustrated through the numerical examples presented in the paper.

2. MATERIAL AND METHOD

This section introduces the proposed method in detail. The mathematical formulation, residual error definition, and implementation procedure are presented.

2.1. The Proposed Method

Primarily, the Chelyshkov polynomials (3) and their derivatives can be presented in matrix forms as outlined in scholarly works [22, 24]:

$$\begin{aligned}
&\gamma(\nu) = \boldsymbol{C}(\nu)\boldsymbol{A} \text{ or } \quad y(\nu) = \boldsymbol{X}(\nu)\Lambda^0\boldsymbol{\mathcal{C}}\boldsymbol{\Lambda}, \quad \Lambda^0 = \boldsymbol{I}, \\
&\gamma^{'}(\nu) = \boldsymbol{C}'(\nu)\boldsymbol{A} = \boldsymbol{X}(\nu)\Lambda^1\boldsymbol{\mathcal{C}}\boldsymbol{A}, \\
&\gamma^{''}(\nu) = \boldsymbol{C}''(\nu)\boldsymbol{A} = \boldsymbol{X}(\nu)\Lambda^2\boldsymbol{\mathcal{C}}\boldsymbol{A}, \\
&\vdots \\
&\gamma^{(\kappa)}(\nu) = \boldsymbol{C}^{(\kappa)}(\nu)\boldsymbol{A} = \boldsymbol{X}(\nu)\Lambda^\kappa\boldsymbol{\mathcal{C}}\boldsymbol{A}, \ \kappa = 0,1,...,\omega
\end{aligned} \tag{5}$$

where

$$\boldsymbol{C}(\nu) = \begin{bmatrix} C_0(\nu) & C_1(\nu) & \cdots & C_\omega(\nu) \end{bmatrix}, \ \boldsymbol{X}(\nu) = \begin{bmatrix} 1 & \nu & \dots & \nu^\omega \end{bmatrix}, \boldsymbol{A} = \begin{bmatrix} a_0 & a_1 & \dots & a_\omega \end{bmatrix}^T$$

$$\Lambda = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & \omega \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{(\omega+1)x(\omega+1)}, \ \boldsymbol{\mathcal{C}} = \begin{bmatrix} \binom{0}{0}\binom{1}{0} & \binom{1}{0}\binom{2}{1} & \binom{2}{0}\binom{3}{2} & \dots & \binom{\omega}{0}\binom{\omega+1}{\omega} \\ 0 & -\binom{1}{1}\binom{3}{1} & -\binom{2}{1}\binom{4}{2} & \dots & -\binom{\omega}{1}\binom{\omega+2}{\omega} \\ 0 & 0 & \binom{2}{2}\binom{5}{2} & \dots & \binom{\omega}{2}\binom{\omega+3}{\omega} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & (-1)^\omega\binom{\omega}{\omega}\binom{2\omega+1}{\omega} \end{bmatrix}_{(\omega+1)x(\omega+1)}$$

By using the matrix relation (4) and collocation points $\nu_\lambda \in [0,1]$ defined by

$$\nu_\lambda = \frac{\lambda}{\omega}, \ \lambda = 0,1,...,\omega$$

we have

$$[\gamma^{(h)}(\nu_\lambda)] = \boldsymbol{X}(\nu_\lambda)\Lambda^h\boldsymbol{\mathcal{C}}\boldsymbol{A} \ , \ \mathrm{h} = 0,1...,\omega$$

or the concise form

$$\boldsymbol{Y}^{(h)} = \boldsymbol{X}\Lambda^h\boldsymbol{\mathcal{C}}\boldsymbol{A}, \ h = 0,1...,\omega \tag{6}$$

where

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}(\boldsymbol{\nu}_0) \\ \boldsymbol{X}(\boldsymbol{\nu}_1) \\ \vdots \\ \boldsymbol{X}(\boldsymbol{\nu}_{\boldsymbol{\omega}}) \end{bmatrix} \quad \text{and} \quad \boldsymbol{Y}^{(h)} = \begin{bmatrix} \boldsymbol{\gamma}^{(h)}(\boldsymbol{\nu}_0) \\ \boldsymbol{\gamma}^{(h)}(\boldsymbol{\nu}_1) \\ \vdots \\ \boldsymbol{\gamma}^{(h)}(\boldsymbol{\nu}_{\boldsymbol{\omega}}) \end{bmatrix}.$$

Similarly, using matrix relation (4) and substituting the collocation point into $\gamma^j(\nu)\gamma^{(h)}(\nu)$ and $\gamma^{(s)}(\nu)\gamma^{(r)}(\nu)$, we obtain the matrix representations, respectively [20, 27]:

$$\begin{bmatrix} \boldsymbol{\gamma}^j(\boldsymbol{\nu}_0)\boldsymbol{\gamma}^{(h)}(\boldsymbol{\nu}_0) \\ \boldsymbol{\gamma}^j(\boldsymbol{\nu}_1)\boldsymbol{\gamma}^{(h)}(\boldsymbol{\nu}_1) \\ \vdots \\ \boldsymbol{\gamma}^j(\boldsymbol{\nu}_N)\boldsymbol{\gamma}^{(h)}(\boldsymbol{\nu}_N) \end{bmatrix} = (\overline{\boldsymbol{Y}})^j\boldsymbol{Y}^{(h)} = \begin{bmatrix} \boldsymbol{\gamma}^j(\boldsymbol{\nu}_0) & 0 & \dots & 0 \\ 0 & \boldsymbol{\gamma}^j(\boldsymbol{\nu}_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \boldsymbol{\gamma}^j(\boldsymbol{\nu}_{\boldsymbol{\omega}}) \end{bmatrix} \begin{bmatrix} \boldsymbol{\gamma}^{(h)}(\boldsymbol{\nu}_0) \\ \boldsymbol{\gamma}^{(h)}(\boldsymbol{\nu}_1) \\ \vdots \\ \boldsymbol{\gamma}^{(h)}(\boldsymbol{\nu}_{\boldsymbol{\omega}}) \end{bmatrix}, \tag{7}$$

$$\begin{bmatrix} \gamma^{(s)}(\nu_0)\gamma^{(r)}(\nu_0) \\ \gamma^{(s)}(\nu_1)\gamma^{(r)}(\nu_1) \\ \vdots \\ \gamma^{(s)}(\nu_\omega)\gamma^{(r)}(\nu_\omega) \end{bmatrix} = (\overline{\boldsymbol{Y}})^{(s)}\boldsymbol{Y}^{(r)} = \begin{bmatrix} \gamma^{(s)}(\nu_0) & 0 & \dots & 0 \\ 0 & \gamma^{(s)}(\nu_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \gamma^{(s)}(\nu_\omega) \end{bmatrix} \begin{bmatrix} \gamma^{(r)}(\nu_0) \\ \gamma^{(r)}(\nu_1) \\ \vdots \\ \gamma^{(r)}(\nu_\omega) \end{bmatrix} \tag{8}$$

Here,

$$\overline{\boldsymbol{Y}} = \overline{\boldsymbol{X}}\,\overline{\boldsymbol{\mathcal{C}}}\,\overline{\boldsymbol{A}} \text{ and } \overline{\boldsymbol{Y}}^{(r)} = \overline{\boldsymbol{X}}\,\overline{\boldsymbol{B}}^{r}\,\overline{\boldsymbol{\mathcal{C}}}\,\overline{\boldsymbol{A}} \tag{9}$$

$$\overline{\boldsymbol{X}} = \begin{bmatrix} \boldsymbol{X}(\nu_0) & 0 & \dots & 0 \\ 0 & \boldsymbol{X}(\nu_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \boldsymbol{X}(\nu_\omega) \end{bmatrix}, \overline{\Lambda} = \begin{bmatrix} \Lambda & 0 & \dots & 0 \\ 0 & \Lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Lambda \end{bmatrix}, \overline{\boldsymbol{\mathcal{C}}} = \begin{bmatrix} \boldsymbol{\mathcal{C}} & 0 & \dots & 0 \\ 0 & \boldsymbol{\mathcal{C}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \boldsymbol{\mathcal{C}} \end{bmatrix}, \overline{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A} & 0 & \dots & 0 \\ 0 & \boldsymbol{A} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \boldsymbol{A} \end{bmatrix}$$

To obtain the fundamental matrix relation, if the collocation points are substituted into Equation (1)

$$\sum_{h=0}^{s_1}\sum_{j=0}^{s_2} P_{hj}(\nu_\lambda)\gamma^{j}(\nu_\lambda)\gamma^{(h)}(\nu_\lambda) + \sum_{r=1}^{s_3}\sum_{s=1}^{s_4} O_{rs}(\nu_\lambda)\gamma^{(s)}(\nu_\lambda)\gamma^{(r)}(\nu_\lambda) = h(\nu_\lambda)$$

and then it is arranged by means of the relations (6) and (7), the following matrix equation is obtained as follows:

$$\left\{ \sum_{h=0}^{s_1}\sum_{j=0}^{s_2} \boldsymbol{P}_{hj}(\overline{\boldsymbol{Y}})^{j}\boldsymbol{Y}^{(h)} + \sum_{r=1}^{s_3}\sum_{s=1}^{s_4} \boldsymbol{Q}_{rs}(\overline{\boldsymbol{Y}})^{(s)}\boldsymbol{Y}^{(r)} \right\} \boldsymbol{A} = \boldsymbol{H}, \tag{10}$$

where

$$\boldsymbol{P}_{hj} = \begin{bmatrix} \boldsymbol{P}_{hj}(\nu_0) & 0 & \dots & 0 \\ 0 & \boldsymbol{P}_{hj}(\nu_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \boldsymbol{P}_{hj}(\nu_\omega) \end{bmatrix}, \boldsymbol{Q}_{rs} = \begin{bmatrix} \boldsymbol{Q}_{rs}(\nu_0) & 0 & \dots & 0 \\ 0 & \boldsymbol{Q}_{rs}(\nu_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \boldsymbol{Q}_{rs}(\nu_\omega) \end{bmatrix}, \boldsymbol{\mathrm{H}} = \begin{bmatrix} h(\nu_0) \\ h(\nu_1) \\ \vdots \\ h(\nu_\omega) \end{bmatrix}$$

By using the relation (5), we can express Equation (9) as follows

$$\Omega\, \boldsymbol{A} = \boldsymbol{\mathrm{H}} \quad or \quad [\Omega;\ \boldsymbol{\mathrm{H}}] \tag{11}$$

such that

$$\Omega = \left[\pi_{pq}\right] = \sum_{h=0}^{s_1}\sum_{j=0}^{s_2} \boldsymbol{P}_{hj}\left(\overline{\boldsymbol{X}}\,\overline{\boldsymbol{\mathcal{C}}}\,\overline{\boldsymbol{A}}\right)^{j} \boldsymbol{X}\Lambda^{h}\boldsymbol{\mathcal{C}} + \sum_{r=1}^{s_3}\sum_{s=1}^{s_4} \boldsymbol{Q}_{rs}\,\overline{\boldsymbol{X}}\,\overline{\Lambda}^{s}\,\overline{\boldsymbol{\mathcal{C}}}\,\overline{\boldsymbol{A}}\boldsymbol{X}\Lambda^{r}\boldsymbol{\mathcal{C}} \tag{12}$$

Equations (10) and (11) define a system of nonlinear algebraic equations for the unknown Chelyshkov coefficients. Using Equation (4), the boundary conditions can also be expressed in matrix form as follows:

$$\boldsymbol{K}_i\boldsymbol{A} = \left[\beta_i\right] \textbf{ or } \left[\boldsymbol{K};\beta_i\right],\ \boldsymbol{i} = 0,1,2,...,\omega-1 \tag{13}$$

where

$$\boldsymbol{K}_i = \begin{bmatrix} \boldsymbol{k}_{i0} & \boldsymbol{k}_{i1} & \dots & \boldsymbol{k}_{iN} \end{bmatrix} = \sum_{h=0}^{\omega-1} \left\{ a_{ih}\boldsymbol{X}(0) + \sum_{k=0}^{K} \boldsymbol{c}_{ik}\boldsymbol{X}(\rho_k) + b_{ih}\boldsymbol{X}(1) \right\} \Lambda^{h}\boldsymbol{\mathcal{C}}.$$

Ultimately, by substituting the final rows of the augmented matrix (10) with those of matrix (12), we obtain the updated augmented matrix denoted as

$$\Omega\ A = \mathbf{H} \quad \text{or} \quad \left[\Omega\ ; \mathbf{H}\right], \tag{14}$$

This matrix represents a nonlinear system of algebraic equations. The resulting nonlinear algebraic system was solved in the Maple environment using the ***solve*** command. Since the system may produce more than one set of coefficients, each candidate solution was evaluated by computing the approximation error for the problem under consideration, and the set yielding the smallest error was selected as the final approximation. Solving system (13) yields the matrix of unknown Chelyshkov coefficients. Subsequently, by inserting these coefficients into (3), the solution, in the form of Chelyshkov polynomials, is derived, and denoted as:

$$\gamma(\nu) \cong \gamma_{\omega}(\nu) = \boldsymbol{X}(\nu)\boldsymbol{\mathcal{C}}\boldsymbol{A}.$$

2.2. Residual Error Analysis

In this section, we introduce a method for efficiently estimating the error in Equation (1). We define the error function for the selected point $\nu = \nu_{\lambda}$ as

$$\varepsilon_{\omega}(\nu_{\lambda}) = \left| \sum_{h=0}^{s_1} \sum_{j=0}^{s_2} P_{hj}(\nu_{\lambda}) \gamma_{\omega}^{j}(\nu_{\lambda}) \gamma_{\omega}^{(h)}(\nu_{\lambda}) + \sum_{r=1}^{s_3} \sum_{s=1}^{s_4} O_{rs}(\nu_{\lambda}) \gamma_{\omega}^{(s)}(\nu_{\lambda}) \gamma_{\omega}^{(r)}(\nu_{\lambda}) - h(\nu_{\lambda}) \right| \cong 0 \tag{15}$$

Here, $\gamma_{\omega}(\nu)$ represents the approximate solution of problem (4)–(5), with $\varepsilon_{\omega}(\nu_{\lambda}) \le 10^{-k}$ (k being a positive integer). The truncation limit ω is increased until the error $\varepsilon_{\omega}(\nu_{\lambda})$ at each point falls below a predefined threshold [28, 29].

Using the mean value theorem, an error bound is derived from the residual function (14) to assess the solution's accuracy [24, 30].

$$\left| \int_0^1 \varepsilon_{\omega}(\nu) d\nu \right| \le \int_0^1 \left| \varepsilon_{\omega}(\nu) \right| d\nu \ = \left| \varepsilon_{\omega}(c) \right| \le \int_0^1 \left| \varepsilon_{\omega}(\nu) \right| d\nu \ \Rightarrow \ \left| \varepsilon_{\omega}(c) \right| \le \int_0^1 \left| \varepsilon_{\omega}(\nu) \right| d\nu = \overline{\varepsilon}_{\omega}, \ c \in [0,1].$$

3. NUMERICAL EXPERIMENTS

This section presents several numerical examples for problems (1) and (2) to illustrate the accuracy and effectiveness of the proposed method.

3.1. Numerical Example 1

Let us consider the following three-point second-order nonlinear ordinary differential equation [31] in [0,1]

$$\gamma''(\nu) + \frac{3}{8}\gamma(\nu) + \frac{2}{1089}(\gamma'(\nu))^2 = -1, \tag{16}$$

with the boundary conditions $\gamma(0) = 0$ and $\gamma(1/3) = \gamma(1)$. The exact solution to the problem is unknown. We suppose the problem has a Chelyshkov polynomial solution

$$\gamma_3(\nu) = \sum_{n=0}^{3} a_n C_n(\nu),$$

where $P_{02}(\nu) = 1, P_{01}(\nu) = 3/8, Q_{11}(\nu) = 2/1089$ and $h(\nu) = -1$. For $\omega = 3$, the collocation points are $\nu_0 = 0,\ \nu_1 = 1/3, \nu_2 = 2/3,$ and $\nu_3 = 1$.

The fundamental matrix representation of the problem is

$$\left\{\boldsymbol{P}_{02}\boldsymbol{X}\Lambda^2\boldsymbol{\mathcal{C}}+\boldsymbol{P}_{10}\boldsymbol{X}\boldsymbol{\mathcal{C}}+\boldsymbol{Q}_{11}\overline{\boldsymbol{X}}\overline{\Lambda}\overline{\boldsymbol{\mathcal{C}}}\overline{\boldsymbol{A}}\boldsymbol{X}\Lambda\boldsymbol{\mathcal{C}}\right\}\boldsymbol{A}=\boldsymbol{H},$$

where

$$\boldsymbol{P}_{02}=\text{Diag}[1\quad 1\quad 1\quad 1],\ \boldsymbol{P}_{10}=\text{Diag}[\tfrac{3}{8}\quad \tfrac{3}{8}\quad \tfrac{3}{8}\quad \tfrac{3}{8}],\boldsymbol{X}(1)=[1\quad 1\quad 1\quad 1],\boldsymbol{X}=[\boldsymbol{X}(0)\quad \boldsymbol{X}(\tfrac{1}{3})\quad \boldsymbol{X}(\tfrac{2}{3})\quad \boldsymbol{X}(1)]^T,$$

$$\boldsymbol{Q}_{11}=\text{Diag}[2/1089\quad 2/1089\quad 2/1089\quad 2/1089],\ \boldsymbol{X}(0)=[1\quad 0\quad 0\quad 0],\boldsymbol{X}(1/3)=[1\quad 1/3\quad 1/9\quad 1/27],$$

$$\boldsymbol{X}(2/3)=[1\quad 2/3\quad 4/9\quad 8/27],\ \boldsymbol{A}=\begin{bmatrix}a_0 & a_1 & a_2 & a_3\end{bmatrix}^T,\ \boldsymbol{F}=\begin{bmatrix}h(0)\\ h(\frac{1}{3})\\ h(\frac{2}{3})\\ h(1)\end{bmatrix},\boldsymbol{\mathcal{C}}=\begin{bmatrix}1 & 2 & 3 & 4\\ 0 & -3 & -12 & -30\\ 0 & 0 & 10 & 60\\ 0 & 0 & 0 & -35\end{bmatrix},$$

$$\overline{\boldsymbol{\mathcal{C}}}=\text{Diag}\begin{bmatrix}\boldsymbol{\mathcal{C}} & \boldsymbol{\mathcal{C}} & \boldsymbol{\mathcal{C}} & \boldsymbol{\mathcal{C}}\end{bmatrix},\Lambda=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 3\\ 0 & 0 & 0 & 0\end{bmatrix}\ \text{and}\ \overline{\Lambda}=\text{Diag}\begin{bmatrix}\Lambda & \Lambda & \Lambda & \Lambda\end{bmatrix}.$$

Equation (12) allows us to compute the matrix form of the initial conditions as

$$\begin{bmatrix}\boldsymbol{K};\beta_i\end{bmatrix}=\begin{bmatrix}1 & 2 & 3 & 4 & ; & 0\\ 0 & -2 & 8/9 & -10/27 & ; & 0\end{bmatrix}.$$

Accordingly, we construct the augmented matrix $\begin{bmatrix}\Omega;\ \mathbf{H}\end{bmatrix}$, and its solution gives the Chelyshkov coefficient matrix:

$$\boldsymbol{A}=\begin{bmatrix}0.213626066649 & \text{-}0.0249929342422 & -0.0558323474439 & 0.000964211041680\end{bmatrix}^T$$

Substituting the Chelyshkov coefficients into Equation (3) gives the approximate solution

$$\gamma_3(\nu)=(2.8e-13)+0.716040640803\,\nu-0.500470811938\,\nu^2-0.0337473864588\,\nu^3.$$

Similarly, additional approximate solutions can be derived utilizing the Chelyshkov polynomials for the cases where ω equals *7* and 10, respectively,

$$\gamma_7(\nu)=-(6.3974845e-13)+0.706567318544\,\nu-0.500458436526\,\nu^2-0.0437274389035\,\nu^3$$
$$+0.0155137811906\,\nu^4+0.000789757126708\,\nu^5-0.000191299912043\,\nu^6-0.00000275520876872\,\nu^7,$$

and

$$y_{10}(\nu)=(2.32154081770e-13)+0.706567299626\,\nu-0.500458436502\,\nu^2-0.0437275115490\,\nu^3$$
$$+0.0155143711474\,\nu^4+0.000787723522743\,\nu^5-0.000187718475357\,\nu^6-0.00000590772051318\,\nu^7$$
$$+0.00000109387547493\,\nu^8-(7.04364437559e-9)\,\nu^9+(1.09786657548e-9)\,\nu^{10}.$$

Table 1. Comparison of the approximate solutions for Numerical Example 1.

v	[12]	[18]	[32,33]	[31]	[34]	Proposed Method
v_1	0.0656	0.0656	0.0656	0.065609977212	0.065609977211	0.0656099772123
v_3	0.1658	0.1658	0.1659	0.165875730299	0.165875730297	0.1658757302986
v_5	0.2236	0.2237	0.2237	0.223694391332	0.223694391330	0.2236943913313
v_7	0.2382	0.2382	0.2282	0.238208824449	0.238208824491	0.2382088244919

v_9	0.2091	0.2093	0.2092	0.209203888236	0.209203888234	0.2092038882354

Table 1 shows a comparison between the approximate solutions obtained by the present method for $\omega = 10$ and those obtained by other methods.

Table 2 compares the maximum absolute errors for $\omega = 10$ obtained by the present method with those from various techniques, including the successive iteration method [31], kernel Hilbert space method [34], Sinc-collocation method [18], Adomian decomposition method [12], optimal homotopy asymptotic method [32], and an efficient method [33].

Table 2. Comparisons of the maximum absolute errors for Numerical Example 1.

ω	**[34]**	**[31]**	ω	**[18]**	ω	**[33]**	ω	**Our**
20	3.4e-008	4.0e-008	20	1.6e-005	8	4.0e-006	7	2.0e-007
40	6.1e-010	1.1e-008	25	1.0e-004	10	5.2e-007	10	2.5e-012

The results shown in these tables indicate that the present method is more effective than the other approaches, and that obtained approximations are expected to be closer to the exact solutions.

3.2. Numerical Example 2

We shall examine the ensuing nonlinear equations featuring variable coefficients, as described in [35], given by:

$$\begin{cases} \nu(1-\nu)\gamma''(\nu) + 6\gamma'(\nu) + 2\gamma(\nu) + \gamma^2(\nu) = h(\nu), \ 0 \le \nu \le 1, \\ \gamma(0) + \gamma\left(2/3\right) = \sinh(2/3), \ \gamma(1) + 0.5\gamma\left(0.8\right) = 0.5\sinh(0.8) + \sinh(1), \end{cases} \tag{17}$$

where $h(\nu) = (2 + \nu - \nu^2 + \sinh\nu)\sinh\nu + 6\cosh\nu.$ The precise solution is expressed as $\gamma(\nu) = \sinh(\nu).$ Here, $P_{02}(\nu) = -(\nu - 1)\nu,\ P_{01}(\nu) = 6, P_{10}(\nu) = 2, P_{20}(\nu) = 1.$ The principal matrix equation corresponding to this problem is:

$$\left\{ \boldsymbol{P}_{02}\boldsymbol{X}\Lambda^2\boldsymbol{C} + \boldsymbol{P}_{01}\boldsymbol{X}\Lambda\boldsymbol{C} + \boldsymbol{P}_{10}\boldsymbol{X}\boldsymbol{C} + \boldsymbol{P}_{20}\overline{\boldsymbol{X}}\,\overline{\boldsymbol{C}}\,\overline{\boldsymbol{A}}\boldsymbol{X}\boldsymbol{C} \right\}\boldsymbol{A} = \boldsymbol{H}.$$

When applying the procedure in the previous example for $\omega = 7$ and 9, we obtain the following approximate solutions:

$$\gamma_7(\nu) = (6.80711367e-10) + 0.999999999776\,\nu - (4.20650e-7)\,\nu^2 + 0.166670922000\,\nu^3$$
$$-0.00001810509624\,\nu^4 + 0.00837304821662\,\nu^5 - 0.000045458541593\,\nu^6 + 0.000221207437136\,\nu^7$$

and

$$\gamma_9(\nu) = (1.157390594e-12) + 0.999999999994\,\nu - (7.96575e-10)\nu^2 + 0.166666677673\,\nu^3$$
$$-(6.999799660e-8)\nu^4 + 0.00833359168581\,\nu^5 - (5.93919762e-7)\nu^6 + 0.000199264774867\,\nu^7$$
$$-(7.163175483e-7)\nu^8 + 0.00000304053517145\,\nu^9.$$

Table 3 displays the results for Example 2, comparing the exact solution with the approximations produced by our approach, Adomian decomposition method [12], the kernel Hilbert space method [35], and a numerical scheme [36].

Table 3. Comparison of the approximate solutions for Numerical Example 2.

v	**[12]**	**[35]**	**Our** $\omega = 7$	**Our** $\omega = 10$
0.08	0.080085360	0.080085	0.080085360	0.080085360
0.24	0.242310644	0.242310	0.242310643	0.242310644
0.40	0.410752325	0.410751	0.410752325	0.410752325
0.48	0.498645505	0.498644	0.498645504	0.498645505
0.64	0.684594227	0.684593	0.684594227	0.684594227
0.72	0.783840477	0.783840	0.783840476	0.783840477

0.80	0.888105982	0.888106	0.888105981	0.888105982
0.96	1.114401793	0.114430	1.114401797	1.114401793

Table 4 juxtaposes the absolute error profiles generated by our proposed technique for $\omega = 7$ and 9 with those produced by alternative methodologies [35, 36].

Table 4. Absolute error function result for Numerical Example 2.

v	[35]	[36]	Our $\omega = 7$	Our $\omega = 9$
0.08	3.0e-008	7.9e-005	4.7e-010	1.0e-012
0.24	2.6e-007	1.9e-005	8.7e-010	1.0e-012
0.40	3.1e-007	7.0e-006	7.6e-010	3.0e-012
0.64	3.0e-007	6.6e-007	6.2e-010	5.0e-012
0.72	5.2e-009	1.7e-006	8.2e-010	4.0e-012
0.96	2.2e-005	2.8e-006	4.1e-009	0

A comparison between Tables 3 and 4 demonstrates the superior numerical accuracy achieved by our method compared to other approaches. Figure 1 visually presents the absolute error functions for our method. Both Figure 1 and Table 4 highlight that the errors decrease as ω increase.

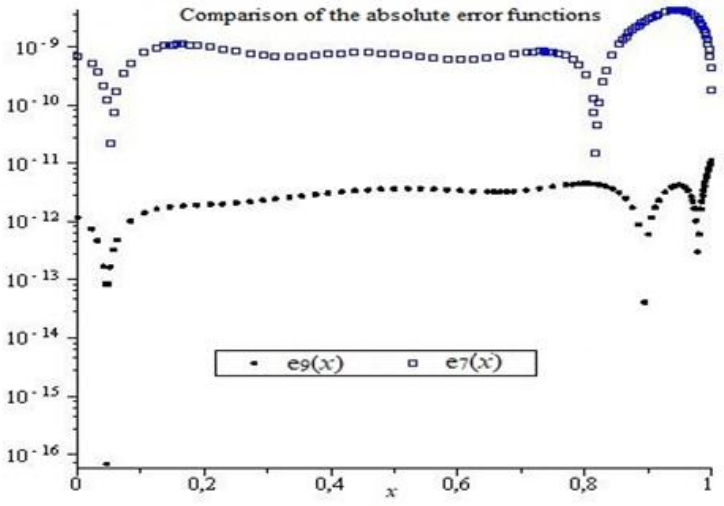


Figure 1. Comparison of Error Functions for Numerical Example 2.

3.3. Numerical Example 3

We solve the following third-order boundary value problem [37]

$$\gamma^{(3)}(\nu) = \nu y''(\nu) + y'(\nu) + \nu\gamma(\nu) + \gamma^2(\nu) - 2\cos(\nu) - \sin^2(\nu), \tag{18}$$

subject to boundary conditions

$$\gamma(1/4) = \sin(1/4),\ \gamma'(1/4) = \cos(1/4),\quad \gamma(1/2) - \gamma(3/4) = \sin(1/2) - \sin(3/4).$$

For problem (18), the exact solution is $\gamma(\nu) = \sin(\nu)$. We now construct the fundamental equation from (11)

$$\left\{ \boldsymbol{P}_{03}\boldsymbol{X}\Lambda^3\boldsymbol{\mathcal{C}} - \boldsymbol{P}_{02}\boldsymbol{X}\Lambda^2\boldsymbol{\mathcal{C}} - \boldsymbol{P}_{01}\boldsymbol{X}\Lambda\boldsymbol{\mathcal{C}} - \boldsymbol{P}_{10}\boldsymbol{X}\Lambda\boldsymbol{\mathcal{C}} - \boldsymbol{P}_{20}\overline{\boldsymbol{X}}\overline{\boldsymbol{\mathcal{C}}}\overline{\boldsymbol{A}}\boldsymbol{X}\boldsymbol{\mathcal{C}} \right\} \boldsymbol{A} = \boldsymbol{H}.$$

Table 5. Absolute error function result for Numerical Example 3.

v	Exact Value	$\omega = 4$	$\omega = 8$	$\omega = 16$
0.1	0.099833	0.099889	0.0998334	0.0998334
0.3	0.295520	0.295526	0.2955202	0.2955202
0.5	0.479425	0.479566	0.4794255	0.4794255

0.7	0.644217	0.644428	0.6442176	0.6442176
0.9	0.783326	0.782747	0.7833269	0.7833269

Upon incorporating the additional conditions into the fundamental equation, we present the outcomes in Table 5 for $\omega = 4, 8$ and 16.

Table 6 furnishes a comparison of the maximum absolute errors across varying numbers of collocation points ω. Furthermore, Figure 2 visually represents the approximations and absolute error functions for $\omega = 4, 8$ and 16.

Table 6. Maximum absolute errors compared for Numerical Example 3.

ω	[37]	ω	**Proposed Method**
4	1.6e-005	4	1.7e-003
16	2.5e-006	8	3.8e-008
64	1.9e-007	16	1.4e-011

A comparison of Tables 5 and 6 demonstrates that our results outperform those from an efficient numerical algorithm [37].

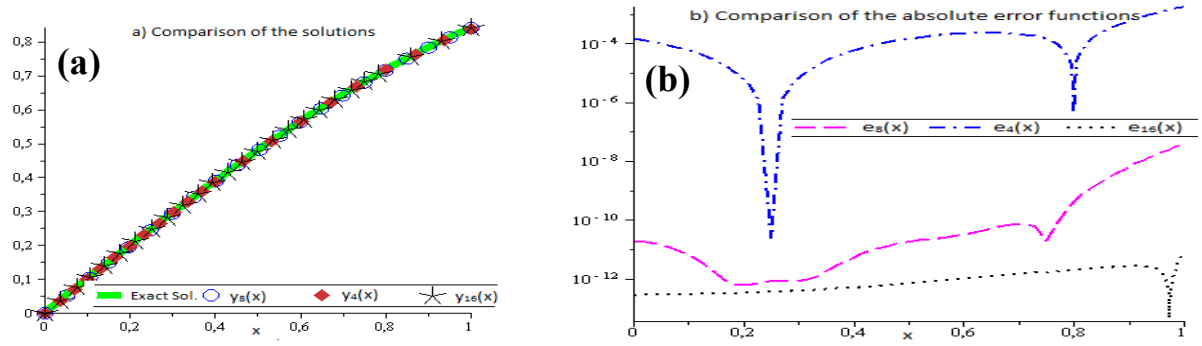


Figure 2. (a) Comparison of the solutions; (b) analysis of the error functions.

Figure 2 underscores the trend of diminishing absolute error values as ω increases.

3.3. Numerical Example 4

Lastly, we consider the following problem [38]

$$\gamma^{(4)}(\nu) + \nu\gamma^2(\nu) + \gamma^3(\nu)\gamma^{(2)}(\nu) - \nu\gamma^{(2)}(\nu)\gamma'(\nu) - \nu^2\gamma^{(4)}(\nu)\gamma^{(3)}(\nu) = h(\nu), \tag{19}$$

with the boundary conditions

$$\gamma(1) = 1,\ \gamma''(0.5) = 3,\ \gamma'''(0.25) = 6,\ \gamma(0) = 0$$

on the interval [0,1]. Here, $h(\nu) = \nu^9 - 48\nu^6 + 2\nu^4 - 576\nu^3 + 24$. Using the proposed technique with $\omega = 4$, we get $\gamma_4(\nu) = \nu^4$, which is the exact solution to the equation (19).

4. RESULTS AND DISCUSSION

The numerical results obtained by the Chelyshkov collocation method are consistent with the exact or reference solutions available in the literature. As shown in the error tables, the approximations exhibit only small deviations for all examples considered. The graphical comparisons further corroborate this observation, showing that the numerical curves nearly coincide with the reference solutions throughout the interval.

It is observed that increasing the polynomial order improves the accuracy of the approximation. However, this expands the associated algebraic system and increases the computational workload. For this reason, choosing a moderate value of N appears to offer a good balance between accuracy and computational cost.

Overall, the examples suggest that the proposed method provides a reliable and stable way to handle nonlinear multi-point boundary value problems. The same framework can also be adapted to similar nonlinear models arising in various applied fields

Statement of Research and Publication Ethics

The study complies with research and publication ethics.

Artificial Intelligence (AI) Contribution Statement

Artificial intelligence tools (such as ChatGPT) were used only to check grammar and improve sentence fluency. The intellectual content, analysis, and conclusions of this study are solely the author's original work.

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